A generalized method for characterizing elastic anisotropy in solid living tissues

J. L. KATZ
Dean of Engineering, Case Western Reserve University, Cleveland, Ohio 44106, USA
A. MEUNIER
Laboratoire de Recherches Orthopédiques, Faculté de Medecine Lariboisière-Saint-Louis, 75010 Paris, France

The microstructure of cortical bone may exhibit either transverse isotropic or orthotropic symmetry, thus requiring either five or nine independent elastic stiffness coefficients (or compliances), respectively, to describe its elastic anisotropy. Our previous analysis to describe this anisotropy in terms of two scalar quantities for the transverse isotropic case is extended here to include orthotropic symmetry. The new results for orthotropic symmetry are compared with previous calculations using the transverse isotropic analysis on the same sets of anisotropic elastic constants for bone, determined either by mechanical or by ultrasonic experiments. In addition, the orthotropic calculation has been applied to full sets of orthotropic elastic stiffness coefficients of a large variety of wood species. Although having some resemblance to plexiform bone in microstructural organization, there is a dramatic difference in both the shear and the compressive elastic anisotropy between the two materials: wood is at least one order of magnitude more anisotropic than bone.

1. Introduction

Recently we presented a scheme for assessing the degree of elastic anisotropy in cortical bone based on calculating certain linear combinations of both the elastic stiffness coefficients and the elastic compliances for the specific elastic symmetry of the materials being considered [1]. This evolved out of an earlier interest in modelling the anisotropic properties of hydroxyapatite [2] as well as in using ultrasonic wave propagation techniques to study the elastic anisotropy both in human bone [3–5] and in bovine bone [6].

First, to calculate the Voigt moduli for a polycrystalline material from the single crystal (or, in the case of cortical bone, from the microstructural texture based properties; see the footnote in [1]) elastic properties it is necessary to average over all spatial orientations of the single-crystal elastic stiffness coefficients [7]. In this calculation three linear relationships among the stiffness coefficients arise naturally [8]:

$$\begin{aligned} c_{11} + c_{22} + c_{33} &= 3A \\ c_{23} + c_{31} + c_{12} &= 3B \\ c_{44} + c_{55} + c_{66} &= 3C \end{aligned} \qquad (1)$$

Similarly, in order to obtain the Reuss moduli, it is necessary to average over all possible spatial orientations of the single-crystal elastic compliances [9]. This also results in three important linear relationships:

$$\begin{aligned} s_{11} + s_{22} + s_{33} &= 3A' \\ s_{23} + s_{31} + s_{12} &= 3B' \\ s_{44} + s_{55} + s_{66} &= 3C' \end{aligned} \qquad (2)$$

Using these expressions, the Voigt bulk modulus, K^{V}, and shear modulus, G^{V} are given by

$$9K^{\mathrm{V}} = 3A + 6B \qquad 5G^{\mathrm{V}} = A - B + 3C \qquad (3)$$

and the equivalent Reuss bulk and shear moduli, K_{R} and G_{R}, by

$$\begin{aligned} 1/K_{\mathrm{R}} &= 3A' + 6B' \\ 5/G_{\mathrm{R}} &= 4A' - 4B' + 3C' \end{aligned} \qquad (4)$$

It is clear from the method of averaging that the Reuss bulk modulus, K_{R}, is just the inverse of the volume compressibility, $s_{11} + s_{12} + s_{13} + 2(s_{12} + s_{13} + s_{23})$. As such, it is isotropic and an invariant with respect to symmetry. Nine times the Voigt bulk modulus, $9K^{\mathrm{V}}$, has the same general form, albeit in stiffness coefficients rather than in compliances, i.e. $c_{11} + c_{12} + c_{13} + 2(c_{12} + c_{13} + c_{23})$. Thus, it also remains unchanged for the symmetries of interest here.

The Voigt modulus also represents the upper bound on the elastic properties of a multiphase system where the strain is uniform across the interface, whereas the Reuss modulus represents the lower bound on the elastic properties where there is a uniform stress distribution across the interface [10]. Thus, the differences between the respective Voigt and Reuss moduli provide a measure of the compressive and shear anisotropies [1, 8, 11, 12]. For convenience the quantities are put into percentages so that the equations for the percentage compressive and shear elastic anisotropy

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TABLE I Elastic stiffness coefficients for various human and bovine bones; c_{ij}(O) for orthotropic symmetry as measured by several researchers; c_{ij}(TI) for transverse isotropy as computed from Equation 8; along with the respective Voigt and Reuss moduli and compressive and shear anisotropy factors

Experiments (Bone type)		c_{11} (GPa)	c_{22} (GPa)	c_{33} (GPa)	c_{44} (GPa)	c_{55} (GPa)	c_{66} (GPa)	c_{12} (GPa)	c_{13} (GPa)	c_{23} (GPa)	K^{V} (GPa)	K_{R} (GPa)	G^{V} (GPa)	G_{R} (GPa)	A_{c}^{*} (%)	A_{s}^{*} (%)
Van Buskik	O	14.1	18.4	25.0	7.00	6.30	5.28	6.34	4.84	6.94	10.4	9.87	6.34	6.07	2.68	2.19
et al. [19] (bovine femur)	TI	16.3	16.3	25.0	6.65	6.65	4.58	4.94	5.89	5.89	10.4	10.0	6.33	6.11	1.67	1.83
Knets [18]	O	11.6	14.4	22.5	4.91	3.56	2.41	7.95	6.10	6.92	10.1	9.52	4.01	3.43	2.68	7.88
(human tibia)	TI	13.0	13.0	22.5	4.24	4.24	2.12	7.37	6.51	6.51	10.1	9.73	4.02	3.48	1.77	7.20
Van Buskirk	O	20.0	21.7	30.0	6.56	5.85	4.74	10.9	11.5	11.5	15.5	15.0	5.95	5.74	1.59	1.82
and Ashman [17] (human femur)	TI	20.9	20.9	30.0	6.21	6.21	4.65	10.7	11.5	11.5	15.6	15.1	5.96	5.76	1.43	1.71
Maharidge [20]	O	21.2	21.0	29.0	6.30	6.30	5.4	11.7	12.7	11.1	15.8	15.5	5.98	5.82	1.11	1.37
(bovine femur, haversian)	TI	21.1	21.1	29.0	6.30	6.30	5.08	11.1	11.9	11.9	15.6	15.2	5.96	5.82	1.33	1.21
Maharidge [20]	O	22.4	25.0	35.0	8.20	7.10	6.10	14.0	15.8	13.6	18.8	18.1	6.88	6.50	1.84	2.85
(bovine femur, plexiform)	TI	23.7	23.7	35.0	7.65	7.65	5.15	12.1	14.7	14.7	18.5	17.7	6.84	6.57	2.30	2.08

are given by

$$A_c^* (\%) = \frac{100\,K^V - K_R}{K^V + K_R}$$

$$A_s^* (\%) = \frac{100\,G^V - G_R}{G^V + G_R} \qquad (5)$$

Clearly the values of A_c^* (%) and A_s^* (%) start at zero (since $K^V = K_R$ and $G^V = G_R$ for isotropic symmetry) and increase as the anisotropy in the material properties increases.

We applied these equations to the anisotropic elastic stiffness coefficients from a number of different experiments on human and bovine bone as well as to those for hydroxyapatite and flurapatite [1]. However, in all of these examples the materials were treated as if they had transverse isotropic (hexagonal) symmetry, even including those specimens which were reported to have orthotropic (orthorhombic) symmetry. The differences among the five independent coefficients needed to characterize the former fully and the nine independent coefficients needed to characterize the latter fully can be seen by examining the respective stiffness matrices given below.

Transverse isotropic

$$[c_{ij}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \qquad (6)$$

where $c_{66} = (c_{11} - c_{12})/2$

Orthotropic

$$[c_{ij}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \qquad (7)$$

In the orthotropic cases that we treated, the differences between c_{11} and c_{22}, c_{13} and c_{23}, c_{44} and c_{55}, and c_{66} and $(c_{11} - c_{12})/2$ were generally of the order of 10% or less, so that the average of each pair was used as the appropriate value for the transverse isotropic calculations. (Note that using the averages leaves K^V and G^V unchanged.) However, this can bias the calculation in some cases, as can be seen below by comparing the results for some of the same bone specimens in both Table I (treated as transverse isotropic) and Table III (treated as orthotropic). Also, there may be times when the above differences are substantially greater than in the specimens considered here. An independent assessment of the differences in elastic anisotropy between the two symmetries may be important in understanding bone remodelling dynamics such as might occur in ageing or at a bone–implant interface. Finally, there are living materials in which there is no question of the orthotropic symmetry found in their elastic properties. Wood is clearly such a material [13–15].

Because of these considerations, we present here the proper elastic anisotropy calculations for orthotropic symmetry using the Voigt and Reuss moduli calculated by averaging over all nine independent stiffness coefficients [16]. In order to appreciate the differences between the two calculations, the equations for K^V, G^V and K_R, G_R for both symmetries are given in the Appendix.

2. Results

2.1. Bone

Table I lists several sets of elastic stiffness coefficients for bones measured on the basis of orthotropic symmetry. Thus, nine independent coefficients, c_{ij}(O) are provided for each case. Also included are the five transverse isotropic coefficients c_{ij}(TI) obtained from the orthotropic constants, based on the following redundancies introduced due to the assumption of higher symmetry:

$$c_{11}(\mathrm{TI}) = c_{22}(\mathrm{TI}) = [c_{11}(\mathrm{O}) + c_{22}(\mathrm{O})]/2$$

$$c_{33}(\mathrm{TI}) = c_{33}(\mathrm{O})$$

$$c_{44}(\mathrm{TI}) = c_{55}(\mathrm{TI}) = [c_{44}(\mathrm{O}) + c_{55}(\mathrm{O})]/2$$

$$c_{13}(\mathrm{TI}) = c_{23}(\mathrm{TI}) = [c_{13}(\mathrm{O}) + c_{23}(\mathrm{O})]/2$$

$$c_{12}(\mathrm{TI}) = c_{12}(\mathrm{O}) \qquad c'_{12}(\mathrm{TI}) = c_{11}(\mathrm{TI}) - 2c_{66}(\mathrm{TI})$$

$$c'_{66}(\mathrm{TI}) = [c_{11}(\mathrm{TI}) - c_{12}(\mathrm{TI})]/2 \qquad c_{66}(\mathrm{TI}) = c_{66}(\mathrm{O}) \qquad (8)$$

Either c_{12} or c_{66} is adjustable in reducing the orthotropic data to transverse isotropy, since they are linearly dependent through the relationship $c_{66} = (c_{11} - c_{12})/2$; c_{11} is considered to be the independent

TABLE II Mean values and standard deviations for the c_{ij}s measured by Van Buskirk and Ashman [17] at each aspect over the entire length of bone (all values in GPa)

	Anterior	Medial	Posterior	Lateral
c_{11}	18.7 ± 1.7	20.9 ± 0.8	20.1 ± 1.0	20.6 ± 1.6
c_{22}	20.4 ± 1.2	22.3 ± 1.0	22.2 ± 1.3	22.0 ± 1.0
c_{33}	28.6 ± 1.9	30.1 ± 2.3	30.8 ± 1.0	30.5 ± 1.1
c_{44}	6.73 ± 0.68	6.45 ± 0.35	6.78 ± 1.0	6.27 ± 0.28
c_{55}	5.55 ± 0.41	6.04 ± 0.51	5.93 ± 0.28	5.68 ± 0.29
c_{66}	4.34 ± 0.33	4.87 ± 0.35	5.10 ± 0.45	4.63 ± 0.36
c_{12}	11.2 ± 2.0	11.2 ± 1.1	10.4 ± 1.0	10.8 ±1.7
c_{13}	11.2 ± 1.1	11.2 ± 2.4	11.6 ± 1.7	11.7 ± 1.8
c_{23}	10.4 ± 1.4	11.5 ± 1.0	12.5 ± 1.7	11.8 ± 1.1

TABLE III Voigt and Reuss averages and the compressive and shear anisotropy factors calculated from the data of Van Buskirk and Ashman [17] for human femoral bone specimens measured at various levels, Z/L (where Z is measured from hip to knee and L is the length of the femur), along the length and at different aspects

Z/L		K^V (GPa)				K_R (GPa)				G^V (GPa)				G_R (GPa)				A_c^* (%)				A_s^* (%)			
		Ant.	Med.	Post.	Lat.	Ant.	Med.	Post.	Lat.	Ant.	Med.	Post.	Lat.	Ant.	Med.	Post.	Lat.	Ant.	Med.	Post.	Lat.	Ant.	Med.	Post.	Lat.
0.3	O	16.2	16.1	15.9	16.5	15.9	15.6	14.3	15.9	5.41	5.60	5.48	6.20	4.91	5.44	5.31	6.00	1.04	1.71	5.26	2.03	4.86	1.45	1.60	1.59
	TI	15.9	16.2	16.0	16.6	15.4	15.7	14.5	16.0	5.36	5.54	5.50	6.21	4.95	5.54	5.35	6.03	1.50	1.51	4.86	1.80	4.15	0.730	1.32	1.51
0.4	O	14.2	15.5	15.1	15.9	13.8	15.2	14.7	15.5	5.78	6.54	6.50	5.65	5.34	6.28	6.35	5.40	1.28	0.880	1.33	1.59	3.94	2.05	1.14	2.32
	TI	14.2	15.4	15.2	15.9	13.8	15.2	14.8	15.4	5.78	6.53	6.51	5.65	5.37	6.31	6.37	5.42	1.24	0.780	1.14	1.67	3.65	1.73	1.14	2.04
0.5	O	15.4	17.1	17.0	17.0	15.1	16.7	16.0	16.7	6.13	6.05	5.82	5.84	5.78	5.69	5.62	5.71	0.830	1.38	2.95	0.870	2.96	3.07	1.72	1.14
	TI	15.5	17.2	17.0	17.1	15.3	16.8	16.1	16.8	6.16	6.06	5.82	5.85	5.84	5.73	5.64	5.73	0.660	1.27	2.48	0.800	2.68	2.81	1.52	1.03
0.6	O	15.0	15.1	15.2	14.8	14.3	14.8	14.6	14.3	5.17	6.08	6.43	6.02	4.70	5.82	6.04	5.67	2.21	1.17	2.02	1.80	4.70	2.16	3.10	3.02
	TI	14.6	15.2	15.3	14.9	13.7	14.9	14.8	14.5	5.12	6.10	6.45	6.04	4.76	5.84	6.12	5.69	3.15	0.980	1.61	1.44	3.77	2.15	2.57	2.98
0.7	O	13.4	14.6	15.8	14.3	12.9	14.3	15.4	13.6	5.79	6.21	6.43	5.99	5.46	5.93	6.24	5.72	1.87	1.19	1.26	2.48	2.89	2.29	1.44	2.28
	TI	13.7	14.8	15.8	14.8	13.4	14.5	15.5	14.4	5.84	6.24	6.43	6.06	5.59	5.99	6.27	5.83	1.10	0.870	1.07	1.49	2.29	1.98	1.34	2.04
Mean	(O)	14.8	15.7	15.8	15.7	14.4	15.3	15.0	15.2	5.65	6.10	6.13	5.94	5.24	5.83	5.91	5.70	1.44	1.27	2.57	1.75	3.87	2.20	1.80	2.07
S.D.	(O)	1.1	1.0	0.8	1.1	1.1	0.9	0.7	1.2	0.37	0.34	0.46	0.20	0.43	0.31	0.44	0.21	0.58	0.31	1.70	0.60	0.93	0.58	0.76	0.73

term because it is one of the most accurately determined quantities. This is why two methods of computing c_{12} and c_{66} are provided in Equation 8. Thus, the reductions have been computed twice: one set using c_{12}(TI) and c'_{66}(TI) along with the rest of the c_{ij}(TI); one set using c'_{12}(TI) and c_{66}(TI) along with the same c_{ij}(TI) as immediately above.

The elastic anisotropy equations have been applied to both the orthotropic and the transverse isotropic data sets listed in Table I. The results, along with the Voigt and Reuss averages, are also presented in Table I; the average of the two calculations for the transverse isotropic A_c^* (%) and A_s^* (%) can be compared with the respective orthotropic anisotropy factors.

Table II lists the average values and standard deviations for each of the c_{ij}s taken over the full length of bone measured for each aspect (anterior (A), medial (M), posterior (P) and lateral (L)) from the data of [17]. In Table III we present the Voigt and Reuss averages and the compressive and shear anisotropy factors at each level for each aspect for both orthotropic symmetry (O) and transverse isotropy (TI) using the values computed from Equation 8.

2.2. Wood

For wood, Table IV corresponds to Table III for bone. Table IV lists the Voigt and Reuss averages for a number of wood species, along with the corresponding values of the shear and compressive elastic anisotropy factors computed on the basis of orthotropic symmetry alone.

3. Discussion

3.1. Bone

Upon examining Table I it can be seen that with the exception of the shear anisotropy for the Knets [18] data (over 7% in both cases) the rest of the values are quite small (below 3% in all cases). However, there is a relatively large change in the compressive anisotropy for the bones of Van Buskirk *et al.* [19] and Knets [18] in going from the measured orthotropic data to the computation based on the higher symmetry. Since K^V does not vary with respect to the change in symmetry, it is K_R which dominates. The comparatively large differences between c_{11} and c_{22} and between c_{13} and c_{23} co-operate here to produce this result.

However, it is in the analysis of the human femur data of Van Buskirk and Ashman [17] presented in Tables II and III where we are able to appreciate the full significance of these scalar elastic anisotropy factors. In this example we show how they can be used in assessing the relative anisotropy to be found throughout the length and around the various aspects of such a long bone.

When dealing with bone as a transverse isotropic material it is possible to get a rough measure of the degree of anisotropy by examining the ratio of c_{33} to c_{11} for compression and of c_{66} to c_{44} for shear. Thus, it is tempting to extend this simplistic scheme to the orthotropic case. Of course, in this case one would have to consider the ratios of c_{33} to each of c_{11} and c_{22} and of c_{66} to each of c_{44} and c_{55}. Also, the off-diagonal terms c_{12}, c_{13} and c_{23} might also be considered, to see whether these appeared to be anything unusual. However, this can entail considerable difficulties, as the following analysis shows. The equations of anisotropy for the orthotropic case are so involved that a superficial compartmentalization of the set of orthotropic stiffness coefficients into its three subsets – (c_{11}, c_{22}, c_{33}), (c_{44}, c_{55}, c_{66}) and (c_{12}, c_{13}, c_{23}) – and analysing them separately can be quite misleading.

In Table II we have reported the mean and standard deviation of each measured orthotropic stiffness coefficient, c_{ij}(O), within each quadrant for the entire length of bone. Thus, we can test whether there are any significant differences between the corresponding stiffness coefficients from aspect to aspect as we transverse the length of bone. Table V presents those parameters which were significant at $p \leqslant 0.05$ in an analysis of the comparison between the means among each pair of aspects over the entire length of bone using a pairwise difference test on the corresponding c_{ij}s in each subset (c_{11}, c_{22}, c_{33}), (c_{44}, c_{55}, c_{66}) and (c_{12}, c_{13}, c_{23}) as well as on the appropriate ratios of c_{ij}/c_{mn} within each subset. Table V also presents results of the

TABLE IV Voigt and Reuss averages and the compressive and shear anisotropy factors calculated from the data of Bucur [14] for wood specimens from several tree species

Species	K^V (GPa)	K_R (GPa)	G^V (GPa)	G_R (GPa)	A_c^* (%)	A_s^* (%)
*P. abies**	2.38	0.839	0.890	0.0820	47.8	83.2
*P. rubens** 1	3.10	0.984	0.843	0.115	51.8	75.9
*P. rubens** 2	3.55	0.986	1.36	0.127	56.5	83.0
*P. sitchensis** 1	2.30	0.771	1.00	0.127	49.7	77.5
*P. sitchensis** 2	3.05	0.885	1.10	0.184	55.0	71.4
*P. sitchensis** 3	3.21	0.659	1.01	0.117	64.9	79.2
*P. engelmannii**	2.88	0.827	0.764	0.0540	55.3	86.7
*A. pseudoplatanus**	4.11	2.19	1.71	0.996	30.6	26.4
A. platanoides†	5.15	2.15	1.85	1.17	41.1	22.6
A. macrophyllum† 1	2.60	1.32	1.58	0.885	32.7	28.3
A. macrophyllum† 2	2.67	1.16	1.53	0.690	39.3	37.7
A. saccharum† 1	4.60	2.10	1.57	0.662	37.3	40.8
A. saccharum† 2	3.62	1.33	1.76	0.819	46.4	36.5
A. rubrum†	2.37	1.76	1.29	0.728	14.8	27.7

*Resonance spruce (*Picea* spp.).
†Fiddleback maple (*Acer* spp.).

TABLE V Parameters which are significantly different when comparing any two aspects from the Van Buskirk and Ashman [17] data over the entire length of the femur (pairwise difference t-test)

Aspect	Medial	Posterior	Lateral
Anterior	c_{11} (p < 0.03)	–	–
	c_{22} (p < 0.02)	c_{22} (p < 0.05)	c_{22} (p < 0.05)
	c_{66} (p < 0.04)	c_{66} (p < 0.01)	–
	–	c_{23} (p < 0.04)	–
	–	c_{23}/c_{12} (p < 0.04)	–
	A_s^* (p < 0.01)	A_s^* (p < 0.005)	A_s^* (p < 0.01)
Medial	x	None	None
Posterior	x	x	None

same analysis for both the compressive and the shear elastic anisotropy factors.

For the subset (c_{11}, c_{22}, c_{33}) there is a low level of significance between the c_{11}s only for the A–M aspects and between the c_{22}s for the A–M, A–P and A–L; all other paired aspects display no significance. Interestingly, there are no significant differences between any pair of aspects for the ratio c_{11}/c_{13} and c_{22}/c_{33}. The subset (c_{44}, c_{55}, c_{66}) also exhibits very few significant differences; in this case a low level for c_{66} between A–M aspects and a more significant one for c_{66} between the A–P aspects. All other pairs of aspects show no significant differences for all three c_{ij}. Also, there are no significant differences between any pair of aspects for the two ratios c_{44}/c_{66} and c_{55}/c_{66}. Finally, the subset (c_{12}, c_{13}, c_{23}) exhibits the same lack of significant differences between each pair of aspects except low ones between A–P both for the ratio c_{23}/c_{12} and for c_{23} itself.

Clearly, it is neither simple nor straightforward to decide whether any important relationships exist based on examining the c_{ij}s or the c_{ij}/c_{mn} ratios in Table V. Indeed, using only the ratios, one might draw the conclusion that there are probably no significant differences in the elastic anisotropy throughout the bone, since only one of the ratios shows even a small significant difference for only one aspect pair, the A–P.

However, an examination of the elastic anisotropy factors in Table V shows that this is not the case. There are highly significant differences between the anterior aspect and each of the other three aspects in turn for the shear anisotropy case; there are no significant differences for the compressive anisotropy factors. Thus, with respect to shear anisotropy, the anterior quadrant of this femur is significantly different from the other three quadrants, which are not significantly different from each other.

Even if additional c_{ij} or c_{ij}/c_{mn} ratios were found to be significant or, even disregarding the factor of significance, the difficulty of trying to estimate the degree of elastic anisotropy in this term-by-term manner is

TABLE VI Correlation coefficients for A_c^* and A_s^* for various c_{ij}/c_{mn} ratios

	c_{11}/c_{33}	c_{22}/c_{33}	c_{44}/c_{66}	c_{55}/c_{66}	c_{13}/c_{12}	c_{23}/c_{12}
A_c^* (%)	0.46	0.35	0.07	0.06	0.72	0.79
A_s^* (%)	0.55	0.51	0.46	0.15	0.39	0.47

TABLE VII Correlation coefficients for A_c^* and A_s^* for the individual c_{ij}s

	c_{11}	c_{22}	c_{33}	c_{44}	c_{55}	c_{66}	c_{12}	c_{13}	c_{23}
A_c^* (%)	0.40	0.33	0.00	0.02	0.22	0.11	0.43	0.35	0.55
A_s^* (%)	0.60	0.63	0.18	0.21	0.51	0.55	0.32	0.19	0.36

compounded by not knowing the weighting factors to assign to each term. This can be appreciated by examining the multivariate regression analysis on A_c^* (%) and A_s^* (%), respectively, against all six ratios in the correlation coefficients in Table VI, or against the nine individual c_{ij}s in Table VII. The variations in both tables show that an attempt at an *a priori* guess at what distribution of weighting factors should be used to work only with the raw data is really useless in both cases.

Fortunately, it is not necessary to go through an extended analysis of the data in order to come to such a conclusion. An analysis of the A_c^* (%) and A_s^* (%) data on Table III leads to the same conclusion. The values of A_s^* (%) for the anterior specimens at levels 0.3, 0.4 and 0.6 are all outside 1 standard deviation from the mean of all the values. Also, the values of A_s^* (%) at levels 0.5 and 0.7 are also quite high, roughly half a standard deviation higher than the mean. Although there are three other specimens with A_s^* (%) values also approximately half a standard deviation higher than the mean, they are randomly distributed and may represent either real variations only within a given level of the bone or experimental artefacts in sample preparation.

With regard to the A_c^* (%) values there is one high value which stands out, the posterior sample at level 0.3; a second, less significant, value is that for the posterior sample at position 0.5. Again, these probably represent local variations within those levels only. A similar observation can be made with respect to anisotropy values lower than the respective means. These few ($3A_c^*$ (%); $4A_s^*$ (%)) are randomly distributed among the various quadrants and throughout the length, thus representing local fluctuations in bone properties.

It is thus clear that the anterior aspect is essentially different from the other three quadrants throughout the entire length measured, whereas in addition there are a few local variations in the elastic anisotropy between quadrants at the same levels and between various levels for the same aspect.

It is interesting to note that the same set of observations could be made with the data based on the transverse isotropic symmetry coefficients computed from Equation 8. The A_c^* (%) and A_s^* (%) values for the transverse isotropic cases in Table III provide identical variations in high and low anisotropies, as do the full orthotropic cases. Thus, the same variations in elastic anisotropy are obtained even when using the higher symmetry on the basis that the differences between c_{11} and c_{22}, c_{44} and c_{55}, c_{13} and c_{23}, and c_{66} and $(c_{11} - c_{12})/2$ are due to extrinsic factors in the local structural arrangement and not to an inherent change in bone structure requiring a decrease in symmetry from transverse isotropy.

In effect these calculations show that whether one considers the full orthotropic set of data or works only with transverse isotropy for the Haversian bone, the relative elastic anisotropy is almost the same. This is in marked contrast to the degree of anisotropy observed for the various wood species presented in Table IV.

3.2. Wood

It is clear why these wood specimens, with almost no exceptions, are at least one decade more anisotropic than bone for both the shear and the compressive cases. The ratio of c_{33} to c_{11} for the resonance spruces (woods used for the tops of violins) runs from about 6 to 8, whereas for the fiddleback maples (woods used for the backs of the violins) it runs from approximately 2.5 to 4. The ratio of c_{33} to c_{22} runs from about 9 to 14.5 for the former woods and from approximately 4 to 10.5 for the latter. This should be compared with the narrow range found for ultrasonic measurements in human bone of approximately 1.3 to 1.6 for both c_{33}/c_{11} and c_{33}/c_{22}, and in bovine plexiform (lamellar) bone of about 1.5 to 1.6 for the same ratios.

Even more dramatic for the woods are the extremely low values of c_{66} when compared with c_{44} and c_{55}. This is important in determining the shear anisotropy. Rather than being comparable, as is in the case of bone, c_{66} runs from about one-half to one-seventh of c_{44} or c_{55} for the fiddleback maples, whereas for the resonance spruces c_{66} runs from about 1/14 to 1/22 of c_{44} or c_{55}.

It is clear from these figures that although cortical bone is elastically anisotropic, the degree of anisotropy is much smaller than that found in another living anisotropic material, wood, to which bone is often compared; indeed, the logo of the Orthopaedic Research Society and many other orthopaedic groups is the wooden splint supporting a deformed tree. Apparently the transverse stiffness in bone has proven to be an important factor in the function of long bones, whereas trees have evolved requiring more flexibility and less need for such relatively high transverse stiffness.

To reinforce some opening comments, we believe that the use of these scalar measures of elastic anisotropy has a number of significant future uses. The principal use is to provide a measure of the kind of macroscopic changes in elastic properties resulting from the local microstructural resorption and remodelling which occurs with ageing or at an implant–bone interface, etc. Secondly, large deviations of either A_c^* (%) or A_s^* (%) from the usual range of values found for bone is probably a warning of something untoward in the elastic stiffness data, either experimental artefacts or something very unusual in the local bone formation.

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Appendix

The Voigt and Reuss moduli for both transverse isotropic and orthotropic symmetry are given below.

Transverse isotropic

Voigt

$$K^{V} = [2(c_{11} + c_{12}) + 4c_{13} + c_{33})]/9$$

$$G^{V} = [(c_{11} + c_{12}) - 4c_{13} + 2c_{33} + 12(c_{44} + c_{66})]/30 \qquad \text{(A1)}$$

Reuss

$$K_{R} = [c_{33}(c_{11} + c_{12}) - 2c_{13}^2]/(c_{11} + c_{12} - 4c_{13} + 2c_{33})$$

$$G_{R} = 5\{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]c_{44}c_{66}\}/2\{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2](c_{44} + c_{66}) + [c_{44}c_{66}(2c_{11} + c_{12}) + 4c_{13} + c_{33}]/3\} \qquad \text{(A2)}$$

Orthotropic

Voigt

$$K^{V} = [c_{11} + c_{22} + c_{33} + 2(c_{12} + c_{13} + c_{23})]/9$$

$$G^{V} = [c_{11} + c_{22} + c_{33} + 3(c_{44} + c_{55} + c_{66}) - (c_{12} + c_{13} + c_{23})]/15 \qquad \text{(A3)}$$

Reuss

$$K_{R} = \Delta/[c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11} - 2(c_{11}c_{23} + c_{22}c_{13} + c_{33}c_{12}) + 2(c_{12}c_{23} + c_{23}c_{13} + c_{13}c_{12}) - (c_{12}^2 + c_{13}^2 + c_{23}^2)]$$

$$G_{R} = 15/(4\{(c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11} + c_{11}c_{23} + c_{22}c_{13} + c_{33}c_{12}) - [c_{12}(c_{12} + c_{23}) + c_{23}(c_{23} + c_{13}) + c_{13}(c_{13} + c_{12})]\}/\Delta + 3(1/c_{44} + 1/c_{55} + 1/c_{66})) \qquad \text{(A4)}$$

where

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = c_{11}c_{22}c_{33} + 2c_{12}c_{23}c_{13} - (c_{11}c_{23}^2 + c_{22}c_{13}^2 + c_{33}c_{12}^2) \qquad \text{(A5)}$$

Although the Reuss moduli in Equations A4 and A5 are expressed in terms of the stiffness coefficients, the original equations are developed in terms of the elastic compliances (Equations 2 and 4). Thus, the transformation equations between the c_{ij}s and s_{ij}s are often needed; these are presented below for convenience:

$$s_{11} = (c_{22}c_{33} - c_{23}^2)/\Delta \qquad s_{22} = (c_{33}c_{11} - c_{13}^2)/\Delta$$

$$s_{33} = (c_{11}c_{22} - c_{12}^2)/\Delta$$

$$s_{12} = (c_{13}c_{23} - c_{12}c_{33})/\Delta \qquad s_{13} = (c_{12}c_{23} - c_{13}c_{22})/\Delta$$

$$s_{23} = (c_{12}c_{13} - c_{23}c_{11})/\Delta$$

$$s_{44} = 1/c_{44} \qquad s_{55} = 1/c_{55} \qquad s_{66} = 1/c_{66} \qquad \text{(A6)}$$

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